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Sitting on the Domain Walls of $\mathcal{N}=1$ Super Yang-Mills

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ABSTRACT: In pure $\mathcal{N}=1$ supersymmetric Yang-Mills with gauge group $SU(N)$, the domain walls which separate the N vacua have been argued, on the basis of string theory realizations, to be D-branes for the confining string. In a certain limit, this means that a configuration of k parallel domain walls is described by a $2+1$-dimensional $U(k)$ gauge theory. This theory has been identified by Acharya and Vafa as the $U(k)$ gauge theory with 4 supercharges broken by a Chern-Simons term of level N in such a way that 2 supercharges are preserved. We argue further that the gauge coupling of the domain wall gauge theory goes like $g^2 \sim \Lambda/N$, for large N. In the case of two domain walls, we show that the $U(2)$ world-volume theory generates a quadratic potential on the Coulomb branch at two loops in perturbation theory which is consistent with there being a supersymmetric bound state of the two wall system. A mass gap of order Λ/N is generated around the supersymmetric minimum and we estimate the size of the bound-state to be order $\Lambda/\sqrt{N}$. At large distance the potential reaches a constant that can qualitatively account for the binding energy of the two walls even though stringy effects are not, strictly speaking, decoupled.

1. Introduction

The theory of D-branes in string theory has provided a remarkable bridge between string theory and gauge theories. We now understand how in certain situations the fundamental string *is* the confining string of the gauge theory. In this context a D-brane will be object on which the confining string can end. Such a set up was first described in [1]. Witten argued that type IIA D-branes become the QCD D-brane. An explicit construction of the world-volume theory on these QCD D-branes has been provided in [2]. In this case the gauge theory in the $3+1$ flat dimensions is pure $\mathcal{N}=1$ supersymmetric Yang-Mills. D4-branes wrapped on the internal S^2 correspond to 2-brane in $3+1$ which are naturally identified with the domain walls that separate the N vacua of the pure $\mathcal{N}=1$ theory. In fact such objects are known independently of any string theory construction to be BPS objects preserving half the supersymmetries [3]. In brief, the vacua are labeled by the value of the gluino condensate [4, 5]

$$\langle\lambda\lambda\rangle_j = N\Lambda^3 e^{2\pi i j/N} , \tag{1.1}$$

$j=0,1,\ldots,N-1$. The basic domain wall separates the j-th and $j+1$-th vacua. However, there are BPS bound states which separate the j-th and $j+k$-th. The tension of the bound state is determined by a central charge to be [3]

$$T_k = \frac{N^2\Lambda^3}{4\pi^2}\sin\frac{\pi k}{N} . \tag{1.2}$$

$k=1,\ldots,N-1$. Since the domain walls are D-branes, their collective dynamics should—at least for small separations—be described by the lightest modes of the confining string stretched between them. This means that the configuration of k basic parallel walls should be described in terms of a $U(k)$ gauge theory in $2+1$ dimensions. Since the bulk gauge theory has four supercharges, the domain wall theory should preserve half of these. The domain wall theory was identified by Acharya and Vafa [2] as the $\mathcal{N}=2$ supersymmetric $U(k)$ gauge theory with a supersymmetrized Chern-Simons term of level N which only preserves half of the supersymmetries. The Chern-Simons term arises from the N units of RR 2-form flux through the S^2.

The purpose of this paper is to consider the potential energy between two domain walls at a distance X apart and in particular to understand the origin of the binding energy of the two walls within the large N expansion

$$\Delta T = 2T_1 - T_2 = \frac{\pi\Lambda^3}{4N}\Big(1+\mathcal{O}(1/N^2)\Big) . \tag{1.3}$$

In the string picture, the potential can be calculated by considering closed string diagrams with boundaries on the two walls. The topological expansion is basically

an expansion in $1/N$ since we expect $g_s \sim 1/N$. The leading contribution $\mathcal{O}(g_s^0)$ comes from the annulus digram illustrated in Figure 1. In more detail, the annulus

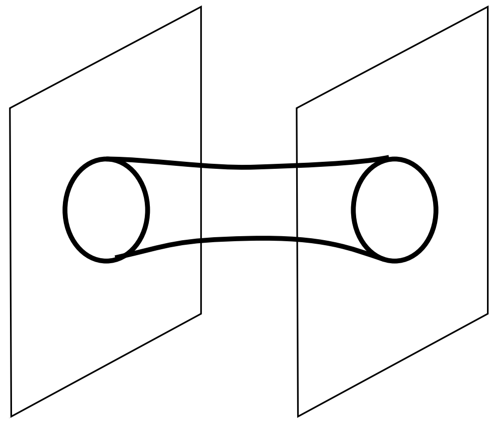

Figure 1: The annulus diagram.

diagram can be viewed either as closed string exchange or a loop of open string. More precisely, there is region of the moduli space where the closed string picture is appropriate and a corresponding region where the open string picture is valid. An explicit cut-off Λ_0, which is usually set around the string scale[1] Λ can be introduced which implements the dichotomy of the diagram. The complete contribution is then a sum

$$V(X) = V^{\text{closed}}(X, \Lambda_0) + V^{\text{open}}(X, \Lambda_0) \tag{1.4}$$

which is independent of Λ_0. We can, following [6], introduce some diagrammatic notation to illustrate the partitioning of the moduli space of a string digram. In this notation an open string is represented by a double line where the Chan-Paton factors can take one of two values corresponding to each of the two walls. A closed string is a wavy line. There are four vertices and their dependence on g_s is illustrated in Figure 2. The annulus diagram splits into two open/closed string graphs realizing the dichotomy (1.4) as illustrated in Figure 3. The effect of the cut-off Λ_0 has the effect of suppressing contributions from higher mass open string modes on the potential so that the leading contribution in the open string sector come from the lightest modes. These modes are captured by the D-brane world-volume gauge theory which in this case is the Acharya-Vafa gauge theory. As the separation of the walls increases, the masses of open strings stretched between the walls increases beyond Λ_0. This means that the contribution to the potential from the open string graphs is suppressed at large X. Actually we shall find that the one-loop contribution to the effective potential of the Acharya-Vafa theory on its Coulomb branch vanishes because of supersymmetry. Consider now the closed string contribution. The closed strings

[1]The string scale is set by the tension of the confining string σ which is expected to be $\sim \Lambda^2$ at large N.

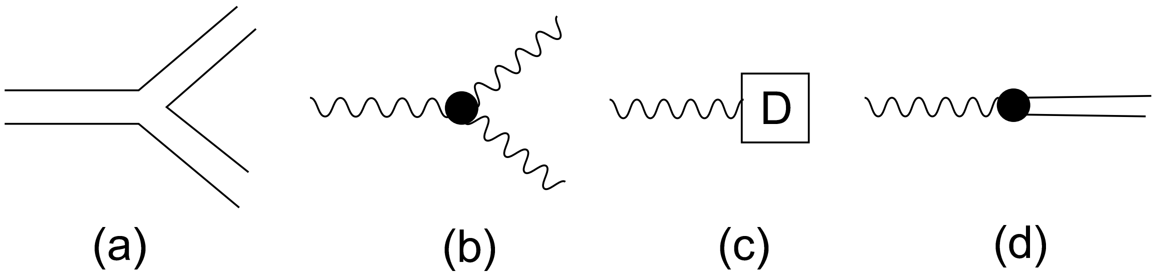

Figure 2: The open/closed string vertices. (a) $g_s^{1/2}$, (b) g_s, (c) g_s^0, (d) $g_s^{1/2}$.

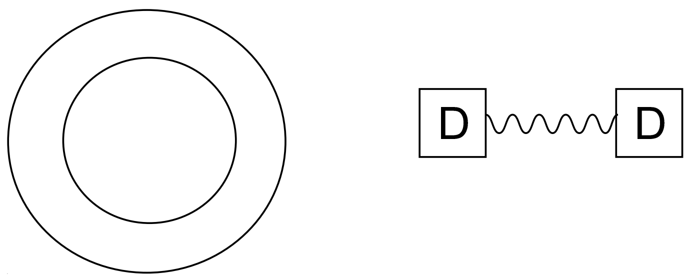

Figure 3: The dichotomy of the annulus into open/closed string graphs.

correspond to glueballs in the gauge theory which are massive since the gauge theory is expected to have a mass gap. This means that there is a long range contribution to the potential from glueball (closed string) exchange of the form

$$V^{\text{closed}}(X) \sim \sum_i \lambda_i e^{-M_i X} \ . \tag{1.5}$$

The effect of the cut-off Λ_0 is to soften the behaviour for $X < \Lambda_0^{-1}$. However, as explained in [9], glueballs always occur in supersymmetric multiplets appropriate to the bulk theory: in this case the $\mathcal{N} = 1$ theory in four dimensions. In other words, in chiral multiplets implying that glueballs always occur in degenerate even and odd parity pairs. The exchange of even and odd parity glueballs precisely cancels since the walls are BPS. This precisely why the force between parallel D-branes in Type II string theory vanishes. For example, the lightest glueball pair are expected to be scalars. The even parity partner couples to the wall tension T and the odd parity partner couples to the wall charge Q. Since the walls are BPS they have $T = |Q|$. In our configuration of two walls, the one on the left has tension/charge $(T, -Q)$ while that on the right (T, Q). So the potential for exchange of the lightest even/odd parity pair glueballs of mass M is

$$V^{\text{closed}}(X) \sim T^2 e^{-MX} + Q(-Q)e^{-MX} = (T^2 - |Q|^2)e^{-MX} = 0 \ . \tag{1.6}$$

This same reasoning extends to the whole tower of glueballs and so the contribution of closed strings to the annulus vanishes. To summarize, we expect the complete annulus contribution to the potential to vanish and this is consistent with the binding energy (1.3) which vanishes at $\mathcal{O}(N^0)$.

The next order in the $1/N$ expansion equates to $\mathcal{O}(g_s)$ corresponding to the "pants diagram" illustrated in Figure (4). In this case, there are five associated

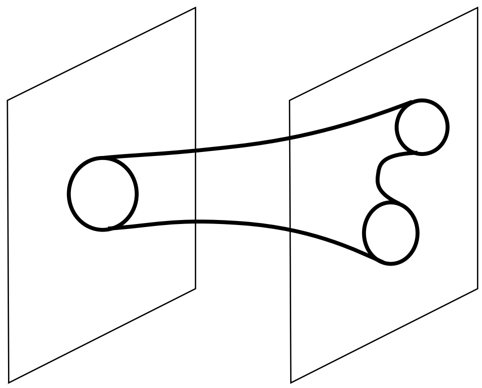

Figure 4: The pants diagram.

open/closed string graphs as illustrated in Figure (5) (essentially a copy of Figure 4 of [6]). The leading contribution to the open string graphs (a) and (b) corresponds to

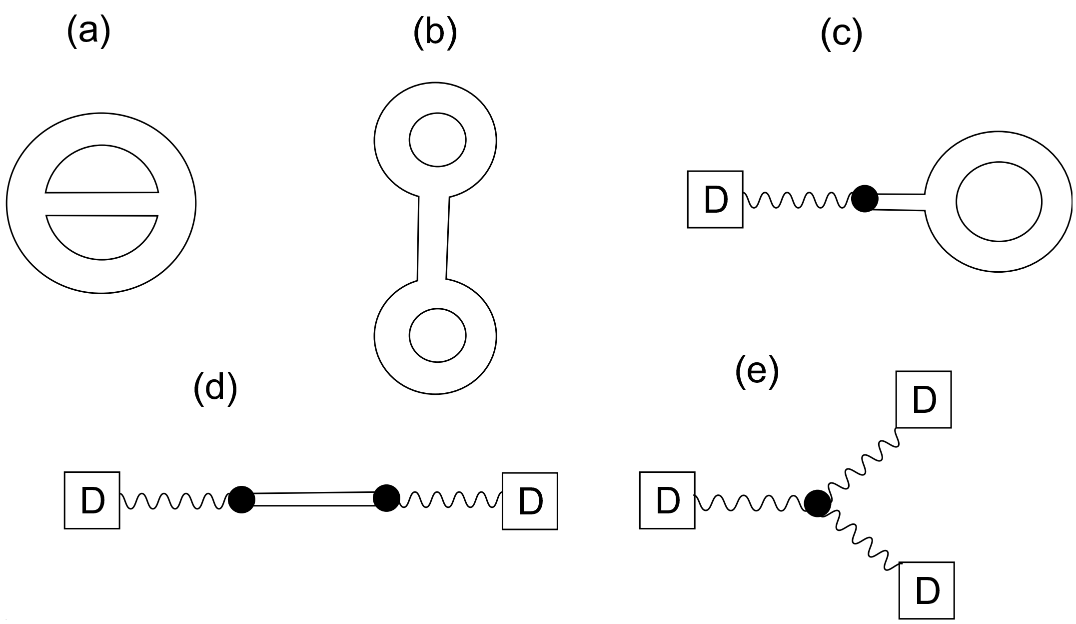

Figure 5: The partitioning of the pants diagram into open/closed string graphs.

the two-loop effective potential of the Acharya-Vafa theory on its Coulomb branch.

The main result of this paper is that this perturbative contribution to two loops is

$$V_{\text{2-loops}}(X) = \frac{2\pi c^6 \Lambda^5 X^2}{N(4\pi^2 c^2 + \Lambda^2 X^2)} , \tag{1.7}$$

where c is determined by the ratio σ/Λ^2 (σ is the tension of the confining string). This result has precisely the right behaviour to account for the existence of a BPS bound-state (it vanishes for $X = 0$) and for the binding energy (the finite limit as $X \to 0$). The question is over what regime we can trust (1.7)? The situation is delicate because some of the fields in the Acharya-Vafa theory have a "topological mass" of order the string scale and so there is no genuine hierarchy between these modes and other excited open string modes. Furthermore, the Acharya-Vafa theory is finite and (1.7) does not need to be regulated. Since there is an explicit cut-off Λ_0 there will be finite cut-off corrections as well as corrections from higher derivative terms in the domain wall DBI action. We expect to be able to trust it for small $X \ll \Lambda$ where the stringy cut-off effects can be neglected. Notice that (1.7) has a minimum at zero which is consistent with the expectation that two walls form a BPS bound state. In addition, around the minimum the theory the potential has the form

$$V_{\text{2-loops}}(X) = \frac{c^4 \Lambda^5 X^2}{2\pi N} + \cdots \tag{1.8}$$

and so the theory has a mass gap $\sim \Lambda/N$. But what about large X where we wish to discover the binding energy? In this limit, the excited open string modes of mass $M \sim \Lambda^2 X$ are suppressed by $\exp(-M^2/\Lambda_0^2)$ [6] but this leaves open the question of cut-off and DBI corrections. We have no definitive argument for why these effects could not alter the asymptotic behaviour of (1.7) for large X:

$$V_{\text{2-loops}}(X) = \frac{2\pi c^6 \Lambda^3}{N} - \frac{8\pi^3 c^8 \Lambda}{N X^2} + \cdots \tag{1.9}$$

The most conservative interpretation is that (1.7) captures the qualitative behaviour of the potential from the open string diagrams (a) and (b). What is clear is that these terms can lead to the constant binding energy even though one normally expects that the open string diagrams are suppressed at large X. In particular, since there are no massless closed string modes we do not expect the power law behaviour $1/X^{2p}$ in the subleading terms to survive stringy corrections.

Now we turn to the two other sub-graphs (c), (d) and (e) in Figure (5). We expect (c) to vanish using the same argument we applied to the annulus: glueballs come in even/odd parity pairs. Contributions (d) and (e), on the other hand, involve an interaction of the glueballs with open string modes. In principle these interactions could be investigated by couplings the Acharya-Vafa theory to the appropriate closed string modes. What is incontrovertible is that the Acharya-Vafa theory breaks the

symmetry between even and odd parity glueballs since the supersymmetry is broken by half on the walls. Consequently, as described in [9], there can be a non-vanishing contribution from glueballs which would have the form

$$V^{(d)+(e)}(X) \sim \frac{1}{N}\sum_i \lambda_i e^{-M_i X} \tag{1.10}$$

for large X.

Using the effective theory, we can estimate the size of the bound state following [6]. The thickness of a single wall was estimated in refs. [7, 8] as $\sim \Lambda/N$. Ignoring factors of order one, near the supersymmetric minimum

$$S_{\text{eff}} \sim N\Lambda^3 \int d^3x \left(-(\partial_i X)^2 - \frac{\Lambda^2}{N^2}X^2 + \cdots \right) . \tag{1.11}$$

The squared size of the bound state is related to the two-point function of X. At leading order in $1/N$ we can ignore the mass term and all the higher interactions in (1.11). At this order X is a massless free field, so

$$\langle X(\epsilon)X(0)\rangle \sim \frac{1}{N\Lambda^3\epsilon} \tag{1.12}$$

This is obviously singular as $\epsilon \to 0$ so in order to make sense of this result as the square of the size of the bound state, we must cut it off at the only natural UV scale in the problem: the string scale $\epsilon \sim \Lambda^{-1}$. The size of the bound-state at leading order in $1/N$ is then $N^{-1/2}\Lambda^{-1}$.

It is interesting to compare the domain wall story above with the discussion in [6] of the bound state of two D-strings and one fundamental string. In this case the $SU(2)$ theory on the two D-branes generates a potential with similar characteristics at the one-loop level. There is a supersymmetric minimum with a mass gap and the potential then rises to a constant. In this case, the one-loop result accounts for all the binding energy of the system at leading order in g_s. One difference is that the power-law behaviour at large X matches the effects of massless closed strings which are absent for the domain wall case.

The paper is organized as follows: in Section 2 we introduce the Acharya-Vafa theory. In section 3 we calculate the Coleman-Weinberg effective potential for the Acharya-Vafa theory on its Coulomb branch up to two loops in perturbation theory.

2. Domain Walls as D-branes

In the Type IIA string theory construction of [2], the fundamental string is the confining string of the gauge theory. The domain walls correspond to D4-branes

wrapped on an S^2 in the internal space; hence they appear as 2-branes in the $3+1$-spacetime of the gauge theory on which the confining string can end. Of course it is rather non-trivial—nay mysterious—that a confining string can end on a domain wall since ordinarily the confining string can only end on fundamental colour charges. In our theory there are only adjoint-valued fields. A heuristic mechanism was described by Witten in [1] (attributed originally to S.-J. Rey). One imagines that the vacua are described by the condensation of the rather elusive QCD monopole, or, for more general vacua, dyon. The basic domain wall separates vacua which differ by the transformation $\theta \to \theta + 2\pi$. In the abelian case, such a transformation shifts the electric charge of the monopole turning it into a dyon. Generalizing this to the non-abelian case one suspects that the corresponding shift in the colour charge is equivalent to that of a quark. A quark excitation can then appear in the domain wall corresponding to a bound-state of a dyon, on one side, and an anti-dyon, on the other. In this way, it is possible for a confining string to end on a domain wall. In another related model, a domain wall can be considered as being composed of a net of baryon vertices, connected by QCD-strings [10]. This model provides an explanation for the N dependence of the domain wall tension, as well as why a string can end on the wall.

We will take as a working hypothesis, the idea that the domain walls are D-branes for the confining string. In this view, the collective dynamics of the domain walls is described in the conventional D-brane-like way by the lightest modes of the confining string which end on the branes. For a configuration of k domain walls the resulting collective dynamics is a Born-Infeld type theory with a $U(k)$ gauge field A_i, a single real adjoint-valued scalar field X which describes the transverse positions of the 2-branes in three-dimensional space. There are also fermions and a particular Chern-Simons term which we describe in detail below.

For the bosonic fields, the domain wall action will be of the form[2]

$$S = \frac{T_k}{k}\int d^3\xi\, \mathrm{STr}\sqrt{-\det\big(\eta_{ij} + D_iXD_jX + \sigma^{-1}F_{ij}\big)} + \cdots\ , \tag{2.1}$$

where the ellipsis represents fermionic terms and also—most importantly—a Chern-Simons term. In the above, σ is the tension of the confining string. The normalization of the action has been fixed by requiring that when the walls are in the ground state the tension is T_k as in (1.2). In principle, this pre-factor should be determined by the way the D4 branes are wrapped on the internal space. At low energies we can expand the square root in the fluctuations to quadratic order:

$$S = T_kV_3 + \frac{T_k}{k\sigma^2}\int d^3\xi\, \mathrm{Tr}\big(-\tfrac{1}{2}(D_i\phi)^2 - \tfrac{1}{4}(F_{ij})^2 + \cdots\big)\ , \tag{2.2}$$

[2]The following expression involves the symmetrized trace, which for our purposes can be replaced by an ordinary trace in the low-energy limit.

where $\phi = \sigma X$ is a real scalar of mass dimension 1. The constant term proportional to the world volume V_3 is just the contribution from the tension of the walls in the ground state. The effective gauge coupling of the theory is evidently

$$g^2 = \frac{k\sigma^2}{T_k} \ . \tag{2.3}$$

For large N (and fixed k), we have $\sigma = c\Lambda^2$ (for a numerical constant c) while $T_k = kN\Lambda^3/(4\pi)$, and so

$$g^2 \underset{N\to\infty}{=} \frac{4\pi c^2\Lambda}{N} \ . \tag{2.4}$$

Notice that in the string theory interpretation $g^2 \sim \sigma^{1/2} g_s$. In other words, the string coupling $g_s \sim 1/N$ as one expects.

We now turn our attention to the fermionic sector. Generically one would expect the D4-branes wrapped on S^2 in the Type IIA set-up to preserve four supercharges on their world-volume. It will ultimately turn out that this is not the case, but for the moment suppose it were true. Then the theory on the 2-branes would then be the $\mathcal{N}=2$ supersymmetric gauge theory in $2+1$ dimensions. This can be considered as the dimensional reduction of an $\mathcal{N}=1$ supersymmetric $U(k)$ gauge theory from four dimensions to three dimensions. We use Wess and Bagger conventions in four dimensions and then dimensionally reduce by removing dependence on x^2; so $\xi^i = (x^0, x^1, x^3)$. The component A^2 is identified with the scalar ϕ. The complex Weyl fermion λ_α can be split into a real and imaginary part in three dimensions:

$$\lambda_\alpha = \frac{1}{\sqrt{2}}(\chi_\alpha + i\psi_\alpha) \ , \qquad \bar{\lambda}_{\dot{\alpha}} = \frac{1}{\sqrt{2}}(\chi_\alpha - i\psi_\alpha) \ . \tag{2.5}$$

Note that in three dimensions one doesn't distinguish between a dotted and un-dotted spinor index. The Lagrangian of the $\mathcal{N}=2$ theory is then[3]

$$\mathcal{L}_{\mathcal{N}=2} = \frac{1}{2g^2}\mathrm{Tr}\big(-(D_i\phi)^2 - \tfrac{1}{2}(F_{ij})^2 - i\chi \not{D}\chi - i\psi \not{D}\psi - 2\chi[\phi,\psi]\big) \ . \tag{2.6}$$

The key observation of Acharya and Vafa [2] is that since there are N units of RR 2-form flux through the internal S^2 on which the D4-branes are wrapped, a Chern-Simons term is induced in the 2-brane world-volume dynamics which breaks the supersymmetry by half. In more detail, for the case of a single domain wall, there is an interaction of the form

$$\int \tilde{A}\wedge F\wedge F = \int \tilde{F}\wedge A\wedge dA \tag{2.7}$$

on the $D4$-brane world-volume, where $\tilde{A}$ is the bulk RR gauge potential and F is the $U(1)$ field strength on the brane. Since there are N units of RR flux through the S^2,

[3] To be clear, $\chi \not{D}\chi = \chi^\alpha \hat{\sigma}^i_{\alpha\beta} D_i \chi^\beta$ where $\hat{\sigma}^i = (\sigma^0, \sigma^1, \sigma^3)$.

$\int_{S^2} \tilde{F} = N$, and so there is an induced Chern-Simons term of level N in the effective theory in 3-dimensions. Extending this argument to many walls and including the fermions, the full action proposed by Acharya and Vafa is

$$\begin{aligned}\mathcal{L}_{\rm AV} = \frac{1}{2g^2}\mathrm{Tr}\Big(&- (D_i\phi)^2 - \tfrac{1}{2}(F_{ij})^2 - i\chi \not{D}\chi - i\psi \not{D}\psi - 2\lambda[\phi,\psi] \\ &+ N\big(\epsilon_{ijk}(A^i\partial^j A^k + \tfrac{1}{3}A^iA^jA^k) + i\chi\chi\big)\Big) \ .\end{aligned} \tag{2.8}$$

The multiplets of $\mathcal{N}=1$ supersymmetry are then (A_i,χ) and (ϕ,ψ) and two supercharges that survive are obtained from the four supercharges in four dimensions by taking the Grassmann parameter of the supersymmetry transformation to be real.

Classically at least, the theory on the domain walls has a Coulomb branch on which the scalar field ϕ develops a VEV involving a scale that we denote φ. If the Chern-Simons term in (2.8) were absent then the theory would have $\mathcal{N}=2$ supersymmetry and a homomorphic description. The real scalars in the unbroken $U(1)^k$ gauge group naturally combine with the dual photons to form k complex scalars. In this case, any effective potential on the Coulomb branch is determined by a superpotential holomorphic in the complex scalars. Holomorphy then forbids the generation of a potential on the Coulomb branch in perturbation theory. However, a potential is generated by non-perturbative instanton effects giving the classic runaway behaviour first uncovered by Affleck, Harvey and Witten [11]. In the present theory, since we only have $\mathcal{N}=1$ supersymmetry there are no holomorphic indulgences and perturbative contributions to the effective potential are not ruled out. There will also be non-perturbative instanton contributions, however, since both g^{-2} and the Chern-Simons coupling scale as N, these will be exponentially suppressed at large N. In what follows we shall be working for the most part at large N and so we shall not discuss these non-perturbative contributions any further. At finite N, the non-perturbative contribution would be important for small values of the VEVs (short distances).

With this restriction in mind, our goal, therefore, is to investigate the perturbative contribution to the effective potential on Coulomb branch obtained by integrating out all the massive modes of the Acharya-Vafa theory. We shall only consider the case of two domain walls, in which case there is a single VEV φ which gives the separation between the domain walls as φ/σ.

We can already see the connection between the binding energy and perturbation theory by expressing the former in terms of the natural perturbative couplings of the Acharya-Vafa theory. These are the gauge coupling g^2, the “topological mass”

$$m = g^2 N = 4\pi c^2\Lambda\Big(1 + \frac{a_1}{N^2} + \frac{a_2}{N^4} + \cdots\Big) \tag{2.9}$$

as well as the Higgs mass φ. The binding energy (1.3) written in terms of these parameters is

$$\Delta T = \frac{\pi\Lambda^3}{4N}\left(1 + \frac{b_1}{N^2} + \frac{b_2}{N^4} + \cdots\right) = \frac{g^2 m^2}{2^8 c^6}\left(1 + \frac{b_1 g^4}{m^2} + \frac{b_2 g^8}{m^4} + \cdots\right) . \tag{2.10}$$

From this we can identify the leading effect at order $1/N$ as coming from two loops in perturbation theory while the sub-leading terms of order $1/N^{2i+1}$ as coming from $2(i+1)$ loops.

3. The Two Wall Potential in Perturbation Theory

In this section, we consider the 2-loop calculation of the effective potential for the $U(2)$ Acharya-Vafa theory. The overall $U(1)$ factor is decoupled and for the purposes of the following calculation we can consider the $SU(2)$ theory instead. On the Coulomb branch the scalar field ϕ develops a VEV which breaks the gauge symmetry from $SU(2)$ to $U(1)$. Using the basis $\{\frac{1}{2}\tau^a\}$ for $SU(2)$,[4] we will choose a VEV

$$\phi = \phi^a \frac{\tau^a}{2} = \varphi\frac{\tau^3}{2}. \tag{3.1}$$

After the Higgs mechanism and in the presence of the Chern-Simons term, ϕ^3 and ψ^3 remain massless. The other fields are either massive or the would-be Goldstone bosons. We discuss them seriatim:

(1) Gauge bosons. The charged components $A_i^\pm$ of the gauge bosons have a complicated propagator which reflects a mixture between the Higgs effect and the topological mass [12] arising from the Chern-Simons term.[5] In Euclidean space, which we now use throughout, and Landau gauge, the propagator is[6]

$$\Delta_{ij}^{\pm}(p) = \frac{(\delta_{ij} - p_i p_j/p^2)(p^2+\varphi^2) - m\epsilon_{ijk}p_k}{(p^2+m_+^2)(p^2+m_-^2)} , \tag{3.2}$$

where

$$m_\pm = \sqrt{\varphi^2 + m^2/4} \pm m/2 . \tag{3.3}$$

The neutral component A_i^3 has a purely topological mass and the propagator in Landau gauge is

$$\Delta_{ij}^{3}(p) = \frac{(p^2\delta_{ij} - p_i p_j) - m\epsilon_{ijk}p_k}{p^2(p^2+m^2)} . \tag{3.4}$$

[4] We will also use the notation $\phi^\pm = \phi^1 \pm i\phi^2$ for the charged components of a field.

[5] The Chern-Simons term in three dimensions gives gauge bosons a mass, the "topological mass" [12]. The situation with both a Chern-Simons term and spontaneous symmetry breaking has been much discussed in the literature: see, for example, [13, 14] and references therein.

[6] The propagator is diagonal in colour indices.

(2) Scalars. The neutral component ϕ^3 is the massless Higgs field while $\phi^{\pm}$ are the would-be Goldstone Bosons and so are massless in Landau gauge.

(3) Fermions. ψ^3 is massless while χ^3 has mass m arising from the supersymmetrized Chern-Simons term in (2.8). The charged fermions $\psi^{\pm}$ and $\chi^{\pm}$ are mixed via the Yukawa coupling with the Higgs VEV. This creates the mass term

$$\begin{pmatrix}\chi^- & \psi^-\end{pmatrix}\begin{pmatrix} m & -i\varphi \\ i\varphi & 0\end{pmatrix}\begin{pmatrix}\chi^+ \\ \psi^+\end{pmatrix} \ . \tag{3.5}$$

The eigenstates $\tilde{\chi}$ and $\tilde{\psi}$ of mass m_+ and $-m_-$ are related to the original basis via

$$\begin{pmatrix}\chi \\ \psi\end{pmatrix}^{\pm} = \sqrt{\frac{2}{m}}\begin{pmatrix} -m_+^{1/2} & m_-^{1/2} \\ i\varphi m_+^{-1/2} & i\varphi m_-^{-1/2}\end{pmatrix}\begin{pmatrix}\tilde{\chi} \\ \tilde{\psi}\end{pmatrix}^{\pm} \ . \tag{3.6}$$

In addition to these fields and their interactions, we have to add the usual gauge fixing terms and associated ghosts $(\bar{c}, c)$. The vertices are those of a conventional spontaneously broken gauge theory except that the Chern-Simons term (2.8) modifies the momentum dependence of the three gauge vertex to

$$(p_1 - p_2)_k \delta_{ij} + (p_2 - p_3)_i \delta_{jk} + (p_3 - p_1)_j \delta_{ik} - m\epsilon_{ijk} \tag{3.7}$$

in Euclidean space.[7]

The effective potential as a function of the VEV φ (which becomes the field of the low-energy effective action) is obtained by integrating out all the massive modes: that is every field except ϕ^3 and ψ^3. In perturbation theory, the contribution is given by summing all the vacuum graphs with massive fields propagating in the loops. It is straightforward to verify that the one-loop contribution vanishes identically due to the mass degeneracies entailed by supersymmetry. At the two loop level, there are two kinds of vacuum graph; namely, the sunset and the figure-of-eight illustrated in Figure (6) below. Although the theory is finite, each separate graph is divergent and must be regularized. Since we wish to preserve supersymmetry we use the dimensional reduction regularization scheme.[8] If the diagrams are calculated correctly, the poles in $d-3$ cancel to leave a finite result. Below we record the result for each separate graph. The singular part of each graph appears in the combination

$$\log\hat{\mu} \equiv \frac{1}{2}\Big(\frac{1}{3-d} + 1 - \gamma + \log 4\pi\Big) + \log\mu \ , \tag{3.8}$$

[7] In Euclidean space the Chern-Simons term is pure imaginary.

[8] In a nutshell, loop momenta propagate in d dimensions, while the tensor and spinor structure is appropriate to 3 dimensions.

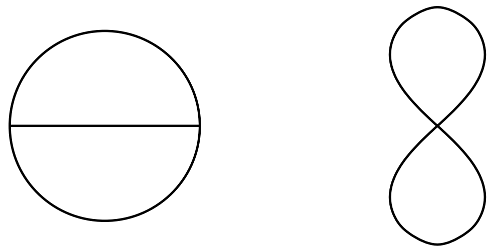

Figure 6: Vacuum graphs at two loops.

where μ is the usual dimensionful parameter of dimensional regularization.

First of all, we have the following sunset graphs:

(1) $\phi^{\pm}$ - χ^3 - $\tilde{\chi}^{\pm}$

$$\begin{aligned}&\frac{g^2(\sqrt{m^2+4\varphi^2}-m)}{32\pi^2\sqrt{m^2+4\varphi^2}}\Big(m^2+m\sqrt{m^2+4\varphi^2}\\&+(5m^2+2\varphi^2+3m\sqrt{m^2+4\varphi^2})\log\frac{2\hat{\mu}}{3m+\sqrt{m^2+4\varphi^2}}\Big)\ .\end{aligned}\tag{3.9}$$

(2) $\phi^{\pm}$ - χ^3 - $\tilde{\psi}^{\pm}$

$$\begin{aligned}&-\frac{g^2(\sqrt{m^2+4\varphi^2}+m)}{32\pi^2\sqrt{m^2+4\varphi^2}}\Big(m^2-m\sqrt{m^2+4\varphi^2}\\&+(-5m^2-2\varphi^2+3m\sqrt{m^2+4\varphi^2})\log\frac{2\hat{\mu}}{m+\sqrt{m^2+4\varphi^2}}\Big)\ .\end{aligned}\tag{3.10}$$

(3) A^3 - $\tilde{\chi}^{\pm}$ - $\tilde{\chi}^{\pm}$

$$-\frac{g^2}{32\pi^2}\Big(m^2-2\varphi^2+m\sqrt{m^2+4\varphi^2}+2(5m^2+4\varphi^2+4m\sqrt{m^2+4\varphi^2})\log\frac{\hat{\mu}}{2m+\sqrt{m^2+4\varphi^2}}\Big)\ .\tag{3.11}$$

(4) A^3 - $\tilde{\psi}^{\pm}$ - $\tilde{\psi}^{\pm}$

$$-\frac{g^2}{32\pi^2}\Big(-3m^2-2\varphi^2+3m\sqrt{m^2+4\varphi^2}+2(5m^2+4\varphi^2-4m\sqrt{m^2+4\varphi^2})\log\frac{\hat{\mu}}{\sqrt{m^2+4\varphi^2}}\Big)\ .\tag{3.12}$$

(5) $A^{\pm}$ - χ^3 - $\tilde{\chi}^{\pm}$

$$\begin{aligned}&-\frac{g^2}{16\pi^2(m^2+4\varphi^2)}\Big(m^4+6m^2\varphi^2+m(m^2-2\varphi^2)\sqrt{m^2+4\varphi^2}\\&+2m(4m^3+16m\varphi^2+(5m^2+4\varphi^2)\sqrt{m^2+4\varphi^2})\log\frac{\hat{\mu}}{2m+\sqrt{m^2+4\varphi^2}}\\&+(m^4+5m^2\varphi^2+4\varphi^4+(5m\varphi^2-m^3)\sqrt{m^2+4\varphi^2})\log\frac{2\hat{\mu}}{3m+\sqrt{m^2+4\varphi^2}}\Big)\ .\end{aligned}\tag{3.13}$$

(6) $A^\pm$ - χ^3 - $\tilde{\psi}^\pm$

$$
\begin{aligned}
&-\frac{g^2}{16\pi^2(m^2+4\varphi^2)}\Big(m^4-10m^2\varphi^2+m(-m^2+6\varphi^2)\sqrt{m^2+4\varphi^2}\\
&-2m(-4m^3-16m\varphi^2+(5m^2+4\varphi^2)\sqrt{m^2+4\varphi^2})\log\frac{\hat{\mu}}{\sqrt{m^2+4\varphi^2}}\\
&+(m^4+5m^2\varphi^2+4\varphi^4-(5m\varphi^2-m^3)\sqrt{m^2+4\varphi^2})\log\frac{2\hat{\mu}}{m+\sqrt{m^2+4\varphi^2}}\Big)\ .
\end{aligned}
\tag{3.14}
$$

(7) A^3 - $\phi^\pm$ - $\phi^\pm$

$$
-\frac{g^2m^2}{16\pi^2}\log\frac{\hat{\mu}}{m}\ .
\tag{3.15}
$$

(8) A^3 - $A^\pm$ - $\phi^\pm$

$$
\begin{aligned}
&-\frac{g^2}{64\pi^2m^2\sqrt{m^2+4\varphi^2}}\Big(14m^3\varphi^2+4m\varphi^4-2m^4\sqrt{m^2+4\varphi^2}\log\frac{\hat{\mu}}{m}\\
&+\varphi^2(5m^3+11m\varphi^2-(5m^2+\varphi^2)\sqrt{m^2+4\varphi^2})\log\frac{\hat{\mu}}{-2m+\sqrt{m^2+4\varphi^2}}\\
&+(m^5+24m^3\varphi^2-22m\varphi^4+(m^4+6m^2\varphi^2)\sqrt{m^2+4\varphi^2})\log\frac{\hat{\mu}}{m+\sqrt{m^2+4\varphi^2}}\\
&+(-m^5+19m^3\varphi^2+11m\varphi^4+(m^4+11m^2\varphi^2+\varphi^4)\sqrt{m^2+4\varphi^2})\log\frac{\hat{\mu}}{3m+\sqrt{m^2+4\varphi^2}}\Big)\ .
\end{aligned}
\tag{3.16}
$$

(9) $A^\pm$ - $A^\pm$ - A^3

$$
\begin{aligned}
&\frac{g^2}{192\pi^2m^2(m^2+4\varphi^2)}\Big(-20m^6-224m^4\varphi^2-224m^2\varphi^4\\
&+(-40m^5+10m^3\varphi^2+12m\varphi^4)\sqrt{m^2+4\varphi^2}\\
&+12m^2(13m^4+56m^2\varphi^2+16\varphi^4-(14m^3-24m\varphi^2)\sqrt{m^2+4\varphi^2})\log\frac{\hat{\mu}}{\sqrt{m^2+4\varphi^2}}\\
&+3(-2m^6-15m^4\varphi^2-29m^2\varphi^4-4\varphi^6\\
&+(2m^5+11m^3\varphi^2+11m\varphi^4)\sqrt{m^2+4\varphi^2})\log\frac{2\hat{\mu}}{-m+\sqrt{m^2+4\varphi^2}}\\
&+(-3m^6+48m^2\varphi^4+(-3m^5-90m^3\varphi^2-66m\varphi^4)\sqrt{m^2+\varphi^2})\log\frac{2\hat{\mu}}{m+\sqrt{m^2+4\varphi^2}}\\
&+(156m^6+672m^4\varphi^2+192m^2\varphi^4+(168m^5+288m^3\varphi^2)\sqrt{m^2+4\varphi^2})\log\frac{\hat{\mu}}{2m+\sqrt{m^2+4\varphi^2}}\\
&+(3m^6+45m^4\varphi^2+135m^2\varphi^4+12\varphi^6\\
&+(-3m^5+57m^3\varphi^2+33m\varphi^4)\sqrt{m^2+4\varphi^2})\log\frac{2\hat{\mu}}{3m+\sqrt{m^2+4\varphi^2}}\Big)\ .
\end{aligned}
\tag{3.17}
$$

(10) $c^{\pm}$ - $c^{\pm}$ - A^3

$$\frac{g^2m^2}{32\pi^2}\log\frac{\hat{\mu}}{m}\ . \tag{3.18}$$

(11) c^3 - $c^{\pm}$ - $A^{\pm}$

$$\begin{aligned}&-\frac{g^2}{32\pi^2\sqrt{m^2+4\varphi^2}}\Big((m^3+3m\varphi^2-(m^2+\varphi^2)\sqrt{m^2+4\varphi^2})\log\frac{2\hat{\mu}}{-m+\sqrt{m^2+4\varphi^2}}\\&-(m^3+3m\varphi^2+(m^2+\varphi^2)\sqrt{m^2+4\varphi^2})\log\frac{2\hat{\mu}}{m+\sqrt{m^2+4\varphi^2}}\Big)\ .\end{aligned} \tag{3.19}$$

The remaining diagrams are figure-of-eights. The only ones that make a non-zero contribution are:

(12) $A^{\pm}$ - A^3

$$\frac{g^2m(m^2+2\varphi^2)}{3\pi^2\sqrt{m^2+4\varphi^2}}\ . \tag{3.20}$$

(13) $A^{\pm}$ - $A^{\pm}$

$$\frac{g^2(m^2+2\varphi^2)^2}{6\pi^2(m^2+4\varphi^2)}\ . \tag{3.21}$$

Summing up all the graphs, the 2-loop contribution to the effective potential is

$$V_{\text{2-loop}}(\varphi)=\frac{g^2m^2\varphi^2}{8\pi^2(m^2+4\varphi^2)}\ . \tag{3.22}$$

Notice that all dependence on the dimensional regularization scale μ disappears as one expects for a finite theory. The other consistency check is that the potential must vanish when $m=0$, since then the theory has $\mathcal{N}=2$ supersymmetry and the effective potential cannot receive perturbative contributions due to holomorphy. The other property in its favour is that the potential is smooth as $\varphi\to 0$. The danger is that as $\varphi\to 0$, the charged components of $(\phi^{\pm},\psi^{\pm})$ becomes massless and have to be included in the low-energy description. But when the VEV vanishes an an $SU(2)$ global symmetry is restored and the potential is then obtained by simply replacing φ with the $SU(2)$ invariant $\sqrt{\varphi^a\varphi^a}$.

4. Discussion

We have shown that the binding energy for domain walls in $\mathcal{N}=1$ Yang-Mills can be accounted for by the gauge theory that describes the collective dynamics of the walls. There are a few unresolved problems. Firstly, we pointed out that the result for

the binding energy from two-loop perturbation theory in the Acharya-Vafa theory could be subject to stringy corrections. Secondly, we only worked to two loops in perturbation theory. Clearly it would be hard to go much beyond this. Since $g^2 \sim \Lambda/N$, higher *odd* loop corrections seem to generate unwanted contributions in even powers in $1/N$ to the binding energy. We believe that these contributions vanish for the following reason: the binding energy is a function of m^2 and thus higher order corrections must run in powers of g^4/m^2. If correct, this is sufficient to rule out contributions from odd number of loops.

We only considered the case of two walls. With many walls we can make the following quick point. The binding energy for k walls at leading order in $1/N$ is

$$\Delta T_k = kT_1 - T_k = \frac{\pi\Lambda^3}{24N}(k^3 - k) + \cdots \ . \tag{4.1}$$

In order to understand the k dependence, we write this as

$$\begin{aligned}\Delta T_k &= \frac{\pi\Lambda^3}{4N}\Big(\frac{k(k-1)(k-2)}{6} + \frac{k(k-1)}{2}\Big) + \cdots \\ &= \frac{\pi\Lambda^3}{4N}\Big(\#\text{triples} + \#\text{pairs}\Big) + \cdots \ .\end{aligned} \tag{4.2}$$

So the k dependence is what one expects from the combinatorics of the pants diagram. Our $U(2)$ result gives the correct coefficient for the sum over pairs; we shall present the calculation for the triples elsewhere.

There are various aspects of the world-volume theory that we postpone for future work. It is potentially interesting to investigate the effect of soft SUSY breaking in the bulk on the world-volume theory. Naïvely one would expect that the world-volume fermions will acquire a mass and hence lead to a non-vanishing one-loop contribution to the vacuum energy, corresponding to the annulus diagram in string theory. Another fascinating aspect of the Acharya-Vafa theory is the possibility of a Seiberg like duality between the $U(k)$ and the $U(N-k)$ world-volume theories. If the world-volume theory describes k walls it should be equivalent in some sense to the theory of $N-k$ (anti-)walls. It would be interesting to see if such a duality could be established.

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Appendix A: Two-Loop Integrals

In this appendix we describe how to evaluate two-loop integrals of the form

$$C(n_1, n_2, n_3) = \mu^{4\epsilon} \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{p_1^{2n_1} p_2^{2n_2} p_3^{2n_3}}{(p_1^2 + m_1^2)(p_2^2 + m_2^2)(p_3^2 + m_3^2)} , \tag{A.1}$$

where $d = 3 - 2\epsilon$ and $p_3 = p_1 + p_2$. The necessary techniques can be found in [15] (see also the useful reference [16]). Using the tricks

$$\frac{p^2}{p^2 + m^2} = 1 - \frac{m^2}{p^2 + m^2} , \qquad \int \frac{d^d p}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{p^a k^b}{k^2 + m^2} = 0 , \tag{A.2}$$

the latter valid in dimensional regularization, one can set up the recursion relation

$$C(n_1, n_2, n_3) = -m_1^2 C(n_1 - 1, n_2, n_3) + (-m_2^2)^{n_2} (-m_3^2)^{n_3} Q_1(n_1 - 1) , \tag{A.3}$$

where

$$Q_1(n) = \mu^{4\epsilon} \int \frac{d^d p_2}{(2\pi)^d} \frac{d^d p_3}{(2\pi)^d} \frac{p_1^{2n}}{(p_2^2 + m_2^2)(p_3^2 + m_3^2)} . \tag{A.4}$$

There are similar recursion relations in p_2 and p_3. By using, (A.2) along with

$$\begin{aligned} R_1(n) &= \mu^{4\epsilon} \int \frac{d^d p_2}{(2\pi)^d} \frac{d^d p_3}{(2\pi)^d} \frac{(p_2 \cdot p_3)^n}{(p_2^2 + m_2^2)(p_3^2 + m_3^2)} \\ &= \begin{cases} \frac{1}{d+n-2} (-m_2^2)^{n/2} (-m_3^2)^{n/2} I(m_2, m_3) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} , \end{aligned} \tag{A.5}$$

where

$$I(m_2, m_3) = \mu^{4\epsilon} \int \frac{d^d p_2}{(2\pi)^d} \frac{d^d p_3}{(2\pi)^d} \frac{1}{(p_2^2 + m_2^2)(p_3^2 + m_3^2)} = \frac{m_2 m_3}{16\pi^2} . \tag{A.6}$$

one has

$$Q_1(n) = \sum_{\substack{a_1, a_2, a_3 \\ a_1 + a_2 + a_3 = n}} \frac{n!}{a_1! a_2! a_3!} 2^{a_3} (-m_2^2)^{a_2} (-m_3^2)^{a_3} R_1(a_3) . \tag{A.7}$$

The recursion relations along with the basic result

$$\begin{aligned} C(0,0,0) &= \mu^{4\epsilon} \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{1}{(p_1^2 + m_1^2)(p_2^2 + m_2^2)(p_3^2 + m_3^2)} \\ &= \frac{1}{16\pi^2} \left(\frac{1}{4\epsilon} - \frac{\gamma}{2} + \frac{1}{2} \log 4\pi + \frac{1}{2} + \log \frac{\mu}{m_1 + m_2 + m_3} \right) , \end{aligned} \tag{A.8}$$

and (A.7) can be used to evaluate all the integrals encountered at two loops.

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